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LARGE CARDINALS IMPLY THAT EVERY REASONABLY DEFINABLE SET OF REALS IS LEBESGUE MEASURABLE

BY

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ABSTRACT

We prove that if there is a supercompact cardinal or much smaller large cardinals, then every set of reals from $L(\mathbf{R})$ is Lebesgue measurable, and similar results. We also introduce some large cardinals.

§1. Introduction

In the old days Solovay proved the, by now classical, result:

1.1. THEOREM. *If* $V^1 = V^{\mathrm{Levi}(\aleph_0, <\kappa)}$ (see below), κ *strongly inaccessible*, then V^1 *satisfies*:

(a) *every set of reals from* $L(\mathbf{R})$ *is Lebesgue measurable*,

(b) *every set of reals from* $L(\mathbf{R})$ *has the property of Baire*,

(c) *in* $L(\mathbf{R})$ *there is no well ordering of the reals*,

(d) *in* $L(\mathbf{R}) \models \omega \rightarrow (\omega)_2^\omega$ *(added by Mathias* [Mt])

(*and on classical results see Solovay* [So]).

For so long some of the central set theorists hoped to prove measurability, etc., of definable sets from large cardinals, and then also the stronger conclusion — *AD*. See Moschovakis [M1] for classical results.

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Magidor [Mg] proved: if there is a measurable cardinal and a precipitous ideal on ω_1, then (a), (b), (c) of 1.1 hold for Σ^1_3 and, from stronger assumptions, Σ^1_4 (precipitous ideals on $\mathscr{P}_{<\aleph_1}(\beth_1)$, $\mathscr{P}_{<\aleph_1}(\beth_3)$). Woodin [W1] proved that "in $L(\mathbf{R})$" every set of reals is Lebesgue measurable, has the Baire property and is not a well ordering of the reals (1.3 (a), (b), (c) for short) by proving $L(\mathbf{R}) \models$ "*AD*" *assuming* there is an elementary embedding $j: L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ with critical point $<\lambda$. This work starts from the thesis that Martin's proof of Π^1_2 determinacy is the right proof.

It did not affect much the theology of the Los Angeles school of descriptive set theorists, by which:

(A) Determinacy has nothing to do with large cardinals [inconsistency of supercompact? "I have no understanding/interest/stake in them"].

(B) Determinacy is much stronger than all known large cardinals.

It also fits well with the theology of general set theorists by which:

(C) Every interesting statement is equi-consistent with some large cardinal.

(D) A consistency result means one from a large cardinal, as those are at least half way to being axioms, not mere statements, hence the experimental observation that they are linearly ordered.

The large majority look at consistency results from *AD* as implications.

But by the L.A. school

(E) *AD* is much more reliable, having inner models.

Anyhow everybody agrees with:

(F) Elementary embeddings come only from large cardinals.

By this theology "there is an $\aleph_2$-saturated normal ideal on ω_1" should be equi-consistent with almost huge — one side was known, the other waits for the inner model.

There was no change by results of Foreman [F], which got results on measurability, etc. from "potent axioms" (saying "large filter" exists on small cardinals), as we do not know much on their consistency strength. This picture was shattered when in Spring 1984 Foreman, Magidor and Shelah [FMS1, §1] was written. From this paper's point of view the important thing was that (relying on [Sh1]):

1.2. THEOREM. *Suppose κ is a supercompact; we can force, without collapsing $\aleph_1$, "D_{ω_1} is $\aleph_2$-saturated"; there is κ-c.c., semi-proper forcing notion P forcing this.*

So some generic elementary embeddings in small cardinals are novel creatures, not extending some old one for large cardinals in an inner model.

In [FMS1]

1.3. Theorem. *If* $MA^+(\aleph_1$*-complete*) (*see* 1.12 *below*) then *any forcing notion preserving stationarity of a subset of* ω_1 *is semi-proper.*

Using the sealing forcing of maximal antichains the following theorem was proved:

1.4. Theorem. *If* $MA^+(\aleph_1$*-complete*) then D_{ω_1} (*the club filter on* ω_1) *is precipitous and even semi-proper*, *subject to the following definition*:

1.5. Definition. Let D be a fine normal filter on $I \subseteq \mathcal{P}(\lambda)$ (our main interest in $I = \omega_1$ or $I \subseteq \{a \subseteq \lambda : |a| < \aleph_1\}$, so we assume it in (2) below).

(1) We say $\{A_i : i < i(*)\}$ (or $\bar{A} = \langle A_i : i < i(*)\rangle$) is a maximal antichain (for D) *if* $A_i \subseteq I$, $A_i \neq \emptyset \mod D$, $[i \neq j \Rightarrow A_i \cap A_j = \emptyset \mod D]$, and $[A \subseteq I \wedge \bigwedge_{i<i(*)} A \cap A_i = \emptyset \mod D \Rightarrow A = \emptyset \mod D]$.

(2) $\bar{A}$ is semi-proper *if*: *for every* large enough cardinal χ and countable $N \prec (H(\chi), \in)$, such that $\bar{A} \in N$, $N \cap \lambda \in I$ *there is* a countable N_1, $N \prec N_1 \prec (H(\chi), \in)$, $N \cap \lambda = N_1 \cap \lambda$ such that $(\exists i < \lg(\bar{A}))[N_1 \cap \lambda \in A_i \wedge i \in N_1]$ (equivalently seal($\bar{A}$) is semi-proper forcing notion; see proof of 2.4).

(3) D is semi-proper *if* every maximal antichain is.

1.6. Remark. When we are interested in uncountable models, the natural restrictions on N, N_1 are more complicated; see [FMS1].

Much parallel activity follows. Foreman, Magidor and Shelah [FMS1], [FMS2] contain some further related theorems, e.g. non-(λ, λ^+)-regular uniform ultrafilters on λ^+ ($\lambda \geqq \aleph_0$ regular) were constructed. [This raises a theological question — who first proved its consistency? See (D), (E) above.] See also the present paper and [Sh2], [Sh3] and [Sh4].

Our result here is

1.7. Theorem. (1) *If there is a supercompact cardinal, then* $L(\mathbf{R})$ *satisfies* (a), (b), (c) *and* (d) *of* 1.1 *holds* (*in fact* $\bigoplus_\kappa$ *holds — see below*).

(2) *In fact much lower cardinals suffice.*

This work was continued by Woodin [W2] who proved (under similar assumptions) that every set in $L(\mathbf{R})$ is represented by a weakly homogeneous tree.

Then Martin and Steel [MS] proved that every set of reals represented by sufficiently weakly homogeneous trees (with measures κ^+-complete) is determined if κ satisfies Definition 4.3, thus deducing $L(\mathbf{R}) \models AD$ from much less than supercompact (relying on the result of [W2] mentioned above for sets in $L(\mathbf{R})$).

This paper is written in the generic way — repeating the steps of its conception, as we feel this has interest *per se*, and as anyhow "everybody" knows the proofs. Naturally there is some overlapping with other works of Spring 1984 in 2.4 with [FMS1], and in 3.4 with [Sh3].

We define here two large cardinals: $\mathrm{Pr}_a(\lambda, f)$, $\mathrm{Pr}_a(\lambda)$ by Shelah (Definition 3.5) and $\mathrm{Pr}_b(\lambda)$ by Woodin — now called a Woodin cardinal.

We also used freely facts which are well known.

Let us mention

1.8. Definition. $\mathrm{Levi}(\lambda, <\mu) = \{f : f$ a two-place function from ordinals to ordinals, $\mathrm{Dom}(f)$ has cardinality $<\lambda$ and $[f(\alpha, \beta) = \gamma \Rightarrow \alpha < \mu \wedge \beta < \lambda \wedge \gamma < \alpha]\}$.

1.9. Definition. $\bigoplus_\kappa$ (for a universe V) will be the statement: in $V^1 = V^{\mathrm{Levi}(\aleph_0, <\kappa)}$ there is an elementary embedding $J : L(\mathbf{R}^V) \to L(\mathbf{R}^{V^1})$ (you can add "with a class of ordinals α, $j(\alpha) = \alpha$", for this paper).

A rephrasing (essentially) of 1.1 is

1.10. Corollary. *If* $\bigoplus_\kappa$, $\kappa > \aleph_1$ *then* (a) + (b) + (c) + (d) *of* 1.1 *holds* (*i.e. for* $L(\mathbf{R}^V)$).

1.11. Definition. (1) $MA_\alpha(\mathrm{Pr})$ means: *if* P is a forcing notion satisfying Pr, D_i a dense subset of P for $i < \omega_1$, $\underset{\sim}{S}_\beta$ a P-name of a stationary subset of ω_1 for $\beta < \alpha$, *then* there is a directed $G \subseteq P$ such that: $G \cap D_i \neq \emptyset$ for $i < \omega_1$, and $\{\zeta < \omega_1 : (\exists\, p \in G) p \Vdash \text{"} i \in \underset{\sim}{S}_\beta \text{"}\}$ is stationary for $\beta < \alpha$.

(2) Let $MA^+(\mathrm{Pr})$ be $MA_1(\mathrm{Pr})$.

$P \lessdot Q$ means complete subforcing, i.e. $P \subseteq Q$ and every maximal antichain of P is a maximal antichain of Q.

§2. The dawn

A conclusion to 1.2, using weakly homogeneous trees, was

2.1. Theorem. *If* κ *is supercompact, then every projective set is Lebesgue measurable and with the property of Baire.*

The second step was

2.2. Conclusion. If κ is supercompact then $\bigoplus_\kappa$ (hence, by 1.11, every set in $L(\mathbf{R})$ is Lebesgue measurable, etc.).

This was obtained by putting together the following simultaneously obtained results:

2.3. Theorem. *If κ is weakly compact, and*
$@^{\mathrm{id}}_\kappa$: *for some κ-c.c. forcing notion P of power κ not adding reals, $\Vdash_P$ "there is a normal $\aleph_2$-saturated ideal $\underset{\sim}{I}$ on ω_1"*
then $\bigoplus_\kappa$ holds.

2.4. Lemma. *For κ-supercompact and $S \subseteq \omega_1$ stationary, co-stationary:*
$@^S$: *there is a κ-c.c. forcing notion P of power, κ not adding reals $\Vdash_P$ "$D_{\omega_1} + S$ is $\aleph_2$-saturated".*

Proof of 2.3. In $V^1 = V^P$ let $Q = \{A \subseteq \omega_1 : A \notin \underset{\sim}{I}\}$ ordered by inverse inclusion, so Q is $\aleph_2$-c.c., i.e. κ-c.c., hence in V, $R \stackrel{\mathrm{def}}{=} P * \underset{\sim}{Q}$ is κ-c.c. Let $R = \bigcup_{i<\kappa} R_i$, R_i increasing continuous, $|R_i| < \kappa$. As κ is weakly compact and R is κ-c.c., $W \stackrel{\mathrm{def}}{=} \{\lambda < \kappa : \lambda \text{ regular}, R_\lambda \lessdot R, R_\lambda \text{ satisfies } \lambda\text{-c.c.}\}$ is stationary, so every real in V^R is in V^{R_λ} for some λ. Now we shall show: In V^R we can rearrange R as Levi$(\aleph_0, <\kappa)$: let $S = \{F : F$ an isomorphism from $R_\lambda (\lambda \in W)$ onto Levi$(\aleph_0, <\lambda)\}$, ordered by inclusion. S as a forcing notion adds no real as W is stationary, so $\mathbf{R}(V^R) = \mathbf{R}(V^{(R*S)})$, but the latter is $R(V^{\mathrm{Levi}(\aleph_0, <\kappa)})$. Now in $V^R = V^{P*\underset{\sim}{Q}}$, $\underset{\sim}{G}_Q$ is a normal (generic) ultrafilter on $\mathscr{P}(\omega_1)^{V^P} = \mathscr{P}(\omega_1)^{V^1}$ so the usual ultrapower $(V^1)^{\omega_1}/\underset{\sim}{G}_Q = M$ (by Mostowski collapse) is defined, as well as the embedding $j : V^1 \to M$. As $\underset{\sim}{I}$ is $\aleph_2$-saturated (in V^P) every Q-name $\underset{\sim}{r}$ of a real (also ordinal) can be represented by $f : \omega_1 \to \omega : \underset{\sim}{r}(n) = l \Leftrightarrow f^{-1}([l]) \in \underset{\sim}{G}_Q$; hence in V^R, $\mathbf{R}(V^R) = \mathbf{R}(M)$. So $j \restriction L(\mathbf{R}^V)$ exemplify $\bigoplus_\kappa$ remembering $\mathbf{R}(V^P) = \mathbf{R}(V)$.

Proof of 2.4. We use RCS iteration $\overline{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leqq \kappa, j < \kappa \rangle$. For each i, if there is in V^{P_i} a maximal antichain $\overline{A} = \langle A^i_\alpha : \alpha < \alpha(*) \rangle$, $|\alpha(*)| > \aleph_1$, of $D_{\omega_1} + S$ (i.e. $\omega_1 - S \subseteq A^i_\alpha \neq \varnothing \bmod D_{\omega_1} + S$, $A^i_\alpha \cap A^i_\beta = \varnothing \bmod D_{\omega_1} + S$) and

$$\begin{aligned} \mathrm{seal}(\overline{A}) = \{(f, C) : \ & f \text{ a function from some countable ordinal} \\ & \alpha + 1 \text{ to } \alpha(*), C \text{ a closed subset of } \alpha + 1, \\ & (\forall \delta \in C \cap S) \delta \in \textstyle\bigcup_{i<\delta} A_{f(i)}\}, \end{aligned}$$

order is defined by:

$$(f_1, C_1) \leqq (f_2, C_2) \Leftrightarrow f_1 \subseteq f_2 \wedge (C_1 = C_2 \cap \mathrm{Dom}\, f_1)\}$$

is semi-proper and Laver's diamond ([L]) gives for i a P_i-name of $\bar{A}$, *then* choose Q_i as such a seal$(\bar{A})$. Otherwise

$$Q_i = \mathrm{Levi}(\aleph_1, (2^{\aleph_1})^{V^{P_i}}).$$

By the $(\omega_1 - S)$-completeness of the Q_i's, reals are not added (by [Sh1], Ch. V).

By the semi-proper iteration lemma (see [Sh1], Ch. X) S is stationary in V^P and also we know P_κ satisfies the κ-c.c. Now suppose $p \Vdash_P$ "$\langle \underset{\sim}{A}_i : i < \kappa \rangle$ is a maximal antichain for $D_{\omega_1} + S$"; as we have used Laver's diamond we have $V^P \models MA^+$ ($\aleph_1$-complete forcing), hence $V^P \models$ "a forcing is semi-proper iff it does not destroy stationary subsets of ω_1", but $Q_{\bar{A}}$ destroys the stationary of no subset of ω_1. By the reflection properties of κ, we could have chosen Q_i as $Q_{\bar{A} \restriction i}$ for many i's; as $\bar{Q}$ was chosen by Laver's diamond we have done it, and then seal$(\bar{A})$ makes the diagonal union of A_j $(j < i)$ contain a club, contradicting $A_i \neq \emptyset \mod D_{\omega_1} + S$.

2.5. Remark. Everybody knows that we can replace in 2.4 P by Levi$(\aleph_1, <\kappa)$ and get an $\aleph_2$-saturated normal ideal on ω_1 (as Levi$(\aleph_1, <\kappa)$ is embeddable into P by a $\lessdot$-embedding; equivalently, in V^P we can find a directed subset of Levi$(\aleph_1, <\kappa)^V$ generic over V).

§3. The dusty road

We may wonder what is the right cardinal in 2.2. Analyzing the proof of MA^+($\aleph_1$-complete) in the case needed in 2.4:

3.1. Theorem. *In* 2.4 *and* 2.2 *we may replace supercompactness by super-strong*:

there is an elementary embedding $j: V \to M$, κ the critical ordinal of j, and $H(j(\kappa)) \subseteq M$ (M a transitive class).

3.1A. Claim. In 2.3 we can replace the use of an $\aleph_2$-saturated ideal by the use of a presaturated one.

In the next stage one may wonder whether we can economize in 2.1, and hopefully find a proof using n cardinals for sets of reals defined by $\underset{\sim}{\Sigma}^1_{n+n_0}$ formulas.

3.2. Lemma. (1) *Suppose $P_0 \lessdot P_1 \lessdot \cdots \lessdot P_n$, P_l a forcing notion adding no reals, P_l not destroying stationary subsets of ω_1 from V or V^{P_m} for $m < l$, and $S \subseteq \omega_1$ stationary, $\Vdash_{P_l}$ "$D_{\omega_1} + S$ is $\aleph_2$-saturated", $\Vdash_{P_l}$ "$\kappa_l = \aleph_2$", $|P_l| < \kappa_{l+1}$, and κ_l is strongly inaccessible.*

Then every Σ^1_{n+2} *set of reals is Lebesgue measurable, Ramsey and has the property of Baire.*

(2) *Suppose we have such* S, P_l $(l \leqq \omega)$ *then* $L(\mathbf{R})$ *satisfies* (a), (b), (c) *and* (d) *of* 1.1.

But when does this occur?

3.3. Definition. (1) Let $(*)^+_\kappa$ mean:

$$\Vdash_{\mathrm{Levi}(\aleph_1,<\kappa)} \text{“}D_{\omega_1} \text{ is semi-proper”.}$$

(2) $(*)_\kappa$ means: letting $P = \mathrm{Levi}(\aleph_0, <\kappa)$, if $\bar{\underset{\sim}{A}} = \langle \underset{\sim}{A}_\alpha ; \alpha < \kappa \rangle$ is a maximal antichain in V^P, then for $\alpha < \kappa$

$$\Vdash_{\mathrm{Levi}(\aleph_1,<\kappa)} \text{“}[\mathrm{Levi}(\aleph_1, <\kappa)/\mathrm{Levi}(\aleph_1, <\alpha)] * \mathrm{seal}(\bar{\underset{\sim}{A}}) \text{ is semi-proper”.}$$

3.3A. Remark. Clearly $(*)^+_\kappa \Rightarrow (*)_\kappa$.

3.4. Lemma. (1) *If* κ *is measurable,* $(*)^+_\kappa$ *holds even after forcing by any* Q, $|Q| < \kappa$ *(semi-proper if you want),* $S \subseteq \omega_1$ *stationary, costationary,* then *for some forcing notion,* P, κ*-c.c. of power* κ *not adding reals,* $|P| = \kappa$, $\Vdash_P$ “$D_{\omega_1} + S$ *is* $\aleph_2$*-saturated*”.

(2) *So if* $\aleph_0 < \kappa_0 < \cdots < \kappa_n$, *each* κ_l *as in* 3.4(1), then *there are* P_l *(for* $l = 0, n$*) as required in* 3.2.

But the assumption in 3.4 is not “a large cardinal axiom”; the problem is not only esthetical — the large cardinal axioms are easily ordered on a scale. So

3.5. Definition. $\mathrm{Pr}_a(\kappa)$ means: $\mathrm{Pr}_a(\kappa, f)$ for every $f : \kappa \to \kappa$, which means: there is $j : V \to M$ (elementary embedding into a transitive class) with critical point κ such that $H(j(f)(\kappa)) \subset M$ and $M^{<\kappa} \subseteq M$.

This property arises from the further analysis of the proof of MA^+($\aleph_1$-complete). (Note $\Vdash_{\mathrm{Levi}(\aleph_1,<\kappa)}$ “MA^+($\aleph_1$-complete)” $\Rightarrow (*)^+_\kappa$.)

Certainly in the large cardinal scale, its use of embedding somewhat deviates.

3.6. Lemma. *If* $\mathrm{Pr}_a(\kappa)$ then κ *is measurable,* $(*)^+_\kappa$; *moreover, if* $|P| < \kappa$ *then* $\Vdash_P$ “$\mathrm{Pr}_a(\kappa)$” *hence* $\Vdash_P$ “$(*)^+_\kappa$ ”

3.7. Conclusion. If there are $\omega + 1$ many cardinals κ, satisfying $\mathrm{Pr}_a(\kappa)$, *then* the conclusions of 1.1 (measurability, Baire and non-well orderings of $\mathbf{R}$) in $L(\mathbf{R})$ hold.

PROOF OF 3.2. (1) Let $Q_j = \{A : A \in V^{P_j},\ A \subseteq \omega_1,\ A \neq \emptyset \mod D_{\omega_1} + S\}$ ordered by inverse inclusion. Note that Q_j is a P_j-name.

Clearly $Q_j \in V^{P_j}$ and by a hypothesis (forcing with P_l preserves stationarity of subsets of ω_1 from V and even from V^{P_m}, $m < l$) we know $Q_j \subseteq Q_{j+1}$.

More still: $P_j * Q_j \lessdot P_{j+1} * Q_{j+1}$ [in V^{P_j}, $D_{\omega_1} + S$ is $\aleph_2$-saturated, so Q_j satisfies the $\aleph_2$-c.c., hence every maximal antichain of Q_j (from V^{P_j}) has cardinality $\leqq \aleph_1$, hence its diagonal union include $\omega_1 - S$ on a club, hence is also a maximal antichain of Q_{j+1}]. Let $G_{P_n} \subseteq P_n$ be generic over V, thus $G_{P_j} \stackrel{\text{def}}{=} G_{P_n} \cap P_j$ is generic over V. Let $G_{Q_n} \subseteq Q_n$ be generic over $V[G_{P_n}]$, then $G_{Q_j} \stackrel{\text{def}}{=} G_{Q_n} \cap Q_j$ is a generic subset of Q_j over $V[G_{P_j}]$. Now in $V[G_{P_j}, G_{Q_j}]$ we can compute the ultrapower $V_j \stackrel{\text{def}}{=} V^{\omega_1}/G_{Q_j}$, where we consider only functions $h : \omega_1 \to V$ which belong to $V[G_{P_j}]$. Set $V_{-1} = V$, and by Los theorem $V_j \prec V_{j+1}$ (before the Moskowski collapse). As in $V[G_{P_j}]$, $D_{\omega_1} + S$ is $\aleph_2$-saturated, it is well known that V_j is well-founded, and we can identify it with a well-founded class in $V[G_{P_j}][G_{Q_j}]$. Also, by $\aleph_2$-saturation of $D_{\omega_1} + S$, we know that every real number which belongs to $V[G_{P_j}][G_{Q_j}]$, in $V[G_{P_j}]$, has a Q_j-name $\underset{\sim}{r}$, hence for some $\langle f_n^r : n < \omega \rangle \in V[G_{P_j}]$, $f_n^r : \omega_1 \to \{0, 1\}$, $\underset{\sim}{r}[G_{Q_j}](n) = 0 \Leftrightarrow (f_n^r)^{-1}(\{0\}) \in G_{Q_j}$ hence $\underset{\sim}{r}[G_{Q_j}] \in V_j$. So $\mathbf{R}^{V_j} = \mathbf{R}^{V[G_{P_j}, G_{Q_j}]}$.

We conclude that

$$\mathbf{R}(V) \prec \mathbf{R}(V[G_{P_0}, G_{Q_0}]) \prec \cdots \prec \mathbf{R}(V[G_{P_n}, G_{Q_n}]).$$

But for each $i < n$, $P_i * Q_i \in V_{\kappa_{i+1}}$, κ_{i+1} is strongly inaccessible in V and all cardinals $< \kappa_{i+1}$ are collapsed to ω in $V[G_{P_{i+1}}, G_{Q_{i+1}}]$. Hence for $i < n$ we can find cardinals $\gamma_{i+1} \in (\kappa_i, \kappa_{i+1})$ and mutual generics $g_{i+1} \subset \mathrm{Levi}(\omega, \gamma_{i+1})$ such that:

$$\mathbf{R}(V[G_{P_0}, G_{Q_0}]) \subset \mathbf{R}(V[g_1]) \subset R(V[G_{P_1}, G_{Q_1}]) \subset \mathbf{R}(V[g_1, g_2])$$
$$\subset \cdots \subset \mathbf{R}(V[g_1, \ldots, g_n]) \subset \mathbf{R}(V[G_{P_n}, G_{Q_n}]).$$

Suppose

$$\varphi(\bar{y}) = \exists x_1 \forall x_2 \cdots Q x_n \psi(\bar{x}, \bar{y})$$

is a Σ^1_{n+2} formula (so ψ is Σ^1_2 or Π^1_2 depending on the parity of n). Then for all $\bar{b}$ from $\mathbf{R}(V[G_{P_0}, G_{Q_0}])$,

$$V[G_{P_n}, G_{Q_n}) \models \varphi[\bar{b}]$$

iff

$$\exists a_1 \in \mathbf{R}(V[g_1]) \forall a_2 \in \mathbf{R}(V[g_1, g_2]) \cdots Q a_n \in \mathbf{R}(V[g_1, \ldots, g_n]) \psi[\bar{a}, \bar{b}]$$

(ψ is either Σ^1_2 or Π^1_2 and so $\psi[a, b]$ is absolute).

Suppose $A \subset \mathbf{R}(V[G_{P_n}, G_{Q_n}])$ is Σ^1_{n+2} in parameters from $\mathbf{R}(V)$. Then by the homogeneity of the Levy collapses and since (c^+) is collapsed to ω in $V[G_{P_0}, G_{Q_0}]$ it follows that $A \cap V[G_{P_0}, G_{Q_0}]$ is Lebesgue measurable, etc. in $V[G_{P_0}, G_{Q_0}]$. But $\mathbf{R}(V[G_{P_0}, G_{Q_0}]) \prec \mathbf{R}(V[G_{P_n}, G_{Q_n}])$ hence $A \cap V[G_{P_0}, G_{Q_0}]$ is the Σ^1_{n+2} set as computed in $V[G_{P_0}, G_{Q_0}]$. Finally $\mathbf{R}(V) \prec \mathbf{R}(V[G_{P_0}, G_{Q_0}])$ and so this Σ^1_{n+2} set computed in V is Lebesgue measurable, etc. in V.

(2) For $i \leqq \omega$ choose G_{P_i}, G_{Q_i} as in the proof of (1). For each $j \leqq \omega$ set $V_j = V^{\omega_1}/G_{Q_j}$ (where we use all functions $h : \omega_1 \to V$ from $V[G_{P_j}]$ and we do not pass to the Mostowski collapse). $V_i \prec V_j$ for $i < j \leqq \omega$.

Let $\underset{\sim}{V} = \bigcup_{i<\omega} V_i$. Thus $\underset{\sim}{V} \prec V_\omega$. V_ω is well founded and so $\underset{\sim}{V}$ is well founded. Let $\underset{\sim}{N} = \operatorname{collapse}(\underset{\sim}{V})$ and let $\underset{\sim}{\kappa} = \sup\{\kappa_i \mid i < \omega\}$. Then there exists an elementary embedding

$$J : V \to \underset{\sim}{N}.$$

Notice $\mathbf{R}(\underset{\sim}{N}) = \bigcup_{i<\omega} \mathbf{R}(N_i) = \bigcup_{i<\omega} \mathbf{R}(V[G_{P_i}, G_{Q_i}])$ where, for $i < \omega$, $N_i = \operatorname{collapse}(V_i)$.

$P_i * Q_i \in V_{\kappa_{i+1}}$ and $P_i * G_i$ collapses all cardinals $< \kappa_i$ to ω. Hence we can force over $V[G_{P_\omega}, G_{Q_\omega}]$ to find $G \subset \operatorname{Levi}(\omega, < \underset{\sim}{\kappa})$ such that G is V-generic and

$$\mathbf{R}(\underset{\sim}{N}) = \bigcup \mathbf{R}(V[G_i])$$

where for $i < \omega$, $G_i = G \cap \operatorname{Levi}(\omega, < \kappa_i)$. Thus $L(\mathbf{R})^{\underset{\sim}{N}}$ can be viewed as a Solovay model obtained from collapsing below $\underset{\sim}{\kappa}$, which is a singular strong limit. Therefore

$$L(\mathbf{R})^{\underset{\sim}{N}} \vDash \text{“Every set is Lebesgue measurable \ldots”}.$$

But there is an elementary embedding

$$J : V \to \underset{\sim}{N},$$

hence

$$L(\mathbf{R}) \vDash \text{“Every set is Lebesgue measurable \ldots”}.$$

Proof of 3.6. The measurability and preservation by small forcing are obvious.

Let us prove $(*)^+_\kappa$. Let $\langle \underset{\sim}{A}_i : i < \kappa \rangle$ be a $\operatorname{Levi}(\aleph_1, < \kappa)$-name of a maximal antichain for D_{ω_1}. Define $f_0 : \kappa \to \kappa$ by: $f_0(i)$ is the minimal cardinal such that for every stationary subset $\underset{\sim}{A}$ of ω_1 from $V^{\mathrm{Levi}(\aleph_1, <\beth_\omega(i))}$, for some $\alpha < f_0(i)$, $\underset{\sim}{A} \cap \underset{\sim}{A}_\alpha$ is stationary (remember $\mathrm{Levi}(\aleph_1, <\kappa)$ satisfies the κ-c.c.); $f_1(i)$ is the minimal cardinal such that $\langle \underset{\sim}{A}_j : j < f_0(i) \rangle$ is a name in $\mathrm{Levi}(\aleph_1, < f_1(i))$; $f_2(i) = \beth_{18}(f_1(i))^+$. Let $j: V \to M$ be as in 3.5 for the function f_2. Let $G_{j(\kappa)} \subseteq \mathrm{Levi}(\aleph_1, < j(\kappa))$ be generic over M, so $G_i \overset{\mathrm{def}}{=} G_{j(\kappa)} \cap \mathrm{Levi}(\aleph_1, < i)$ is generic over V whenever $i < j(f_2)(\kappa)$. In $V[G_\kappa]$, $\langle \underset{\sim}{A}_i[G_\kappa] : i < \kappa \rangle$ is a maximal antichain of D_{ω_1}. Let $<^*$ be a well ordering of $H(2^\kappa)^{V[G_\kappa]}$,

$$S = \{N : N \prec (H(2^\kappa)^{V[G_\kappa]}, \in, <^*),\ N \text{ countable, and there is no } N_1,\ N \prec N_1 \prec (H(2^{2^\kappa}), \in, <^*),\ N_1 \text{ countable, } N_1 \cap \omega_1 = N \cap \omega_1 \in \bigcup_{i \in N_1} A_i[G_\kappa]\}.$$

Clearly in V there exists a $\mathrm{Levi}(\aleph_1, \leqq 2^{2^\kappa})$-name $\langle \underset{\sim}{a}_i : i < \omega_1 \rangle$ such that

$$\Vdash_{\mathrm{Levi}(\aleph_1, < 2^{2^\kappa})} \text{“}\langle \underset{\sim}{a}_i : i < \omega_1 \rangle \text{ is increasing, continuous, each } \underset{\sim}{a}_i \text{ countable, } \bigcup_{i < \omega_1} \underset{\sim}{a}_i = H(2^\kappa)^{V[G_\kappa]}\text{”}.$$

Let $\underset{\sim}{A} = \{i : (\exists N \in S)(\omega_1 \cap \underset{\sim}{a}_i \subseteq |N| \subseteq \underset{\sim}{a}_i)\}$. As $\mathrm{Levi}(\aleph_1, \leqq 2^{2^\kappa})$ preserve stationary sets of $\{b : b \subseteq H(2^\kappa) \text{ is countable}\}$, we have that

$$\Vdash_{\mathrm{Levi}(\aleph_1, \leqq 2^{2^\kappa})} \text{“}\underset{\sim}{A} \text{ is a stationary subset of } \omega_1\text{”}.$$

This holds also in M, so there exists $i < j(f_0)(\kappa)$ such that, in M,

$$\Vdash_{\mathrm{Levi}(\aleph_1, < j(f_1)(\kappa))} \text{“}\underset{\sim}{A} \cap \underset{\sim}{A}_i \text{ is a stationary subset of } \omega_1\text{”}.$$

Therefore (see choice of f_2) this last assertion holds in V too (remember that $H(j(f_2)(\kappa)) \subseteq M$ and note that $S, \underset{\sim}{A} \in M$), so in $M[G_\kappa]$ letting $\kappa(0) = (j(f_1))(\kappa)$, $\kappa(1) = 2^{2^{\kappa(0)}}$, $P_\kappa = \mathrm{Levi}(\aleph_1, < \kappa)$.

$$\begin{aligned} S_1 = \{N \prec (H(\kappa(1))^{M[G_\kappa]}, \in, <^*) :\ & N \text{ countable and for some} \\ & p \in P_{\kappa(0)}/G_\kappa,\ p \text{ is} \\ & (N, P_{\kappa(0)}/G_\kappa)\text{-semi-generic,} \\ & p \Vdash \text{“}N \cap \omega_1 \in \underset{\sim}{A} \cap \underset{\sim}{A}_i\text{”}\} \end{aligned}$$

is a stationary subset of $[H(2^{2^{\kappa(0)}})^{M[G_\kappa]}]^\omega$. But

$$H(2^{2^{\kappa(1)}})^{M[G_\kappa]} = H(2^{2^{\kappa(1)}})^{V[G_\kappa]}$$

hence the same holds with $V[G_\kappa]$ instead of $M[G_\kappa]$.

Now in $V[G_\kappa]$,

$$S_2 = \{N \cap H(\kappa(1)) \colon N \prec H(\beth_3(j(\kappa))^+)^{M[G_\kappa]}, \in, <^*), N \text{ countable } \langle \underset{\sim}{a}_i : i < \omega_1 \rangle, j(\kappa), j \restriction \beth_3(\kappa)^+, A, A_i, \text{ belongs to } N\}$$

is a club of $S_{\leq \aleph_0}(H(\kappa(1)))$; remember $M^{<\kappa} \subseteq M$. So there is $N_2 \in S_2 \cap S_1$, and let N_1 be the Skolem hull of $|N_2|$ in $(H(\beth_3(j(\kappa))^+)^{M[G_\kappa]}$. And let $p \in P_{\kappa(0)}/G_\kappa$ be as in the definition of S_1. So clearly (in $M[G_\kappa]$) p is also $(N_2, P_{f_2(\kappa)})$-semi-generic. So let $\delta \overset{\text{def}}{=} N_1 \cap \omega_1$; as $p \Vdash \delta \in \underset{\sim}{A}$, for some q, $p \leqq q \in p_{\kappa(0)}/G_\kappa$ and $b \in S$,

$$q \Vdash \text{“}\delta \subseteq b \subseteq a_\delta\text{”}.$$

Also, as q is $(N_1, P_{\kappa(0)}/G_\kappa)$-semi-generic, $\langle \underset{\sim}{a}_i : i < \omega_1 \rangle \in N_1$ we have that

$$q \Vdash \text{“}N_1 \cap H(2^\kappa)^{V[G_\kappa]} = \underset{\sim}{a}_\delta\text{”}.$$

Therefore

$$q \Vdash \text{“}b \subseteq N_1 \cap H(2^{2^\kappa})^{V[G_\kappa]}\text{”}$$

so really $b \subseteq N_1$, and thus (see definition of S_2) $j(b) = j''(b) = \{j(x) : x \in b\} \subseteq N_1$, and therefore

$$M[G_\kappa] \models j(b) \in j(S).$$

But N_2, i, q contradict this. This concludes the proof of 3.6.

We can skip 3.4.

3.8. Claim. If $\aleph_0 < \kappa_1 < \kappa_2 < \cdots < \kappa_n$ are strongly inaccessibles and $(*)_{\kappa_l}$ *then* the conclusion of 3.2 holds.

Proof. Let $P_l = \text{Levi}(\aleph_1, \kappa_l)$ for $l = 1, \ldots, n+1$ where we stipulate $\kappa_{n+1} = \beth_8(\kappa_n)$. Let $\underset{\sim}{S}_l$ be the P_{l+1}-name of the diagonal intersection of the diagonal union of every $\bar{A} \in V^{P_l}$, a maximal antichain of D_{ω_1} in V^{P_l}. The union and intersection are well defined modulo D_{ω_1}; as in $V^{P_{l+1}}$ they are of $\aleph_1$ subsets of ω_1. As $(*)_{\kappa_l}$, S_l is a stationary subset of κ_l, with stationary intersection with any stationary $S \in V^{P_l}$ $(S \subseteq \omega_1)$. So we can show inductively $S^* = \bigcap_{l \leq n} S_l \neq \emptyset \bmod D_{\omega_1}$ in $V^{P_{n+1}}$. Now let $Q = \{A \subseteq S^* : A \text{ stationary}\}$ (in $V^{P_{n+1}}$), ordered by inverse inclusion, and proceed as in the proof of 3.2.

3.9. Remark. (1) However, 3.8 is not a big saving, as from the large cardinals for which we can use 3.8 we could also use 3.2 (i.e. get $\aleph_2$-saturated ideals $D_{\omega_1} + S$) (by [Sh 4]).

(2) Here, we can require on j only “$M^\omega \subseteq M$” instead of “$M^{<\kappa} \subseteq M$”.

§4. Happy end

A close look at 3.5 and 3.6 shows their lousiness — $\mathrm{Pr}_a(\kappa)$ implies that there are many smaller cardinals with the desired properties. Further analysis leads to 4.3 below, which is further deviating but totally in the scale of large cardinals.

4.1. Definition. $\mathrm{Pr}_b(\kappa)$, now usually called "κ is a Woodin cardinal", means:

For every $f: \kappa \to \kappa$, there is an elementary embedding $j: V \to M$ with critical point $\lambda < \kappa$, such that $H(j(f)(\lambda)) \subseteq M$ and $f''\lambda \subset \lambda$.

So κ is a Mahlo cardinal, but not necessarily a weakly compact cardinal.

4.2. Theorem. *If* $\mathrm{Pr}_b(\kappa)$ *then* $(*)_\kappa$.

Proof. Let $P \stackrel{\text{def}}{=} \mathrm{Levi}(\aleph_1, <\kappa)$, and $\langle \underset{\sim}{A}_i : i < \kappa \rangle$ be a P-name of a maximal antichain for D_{ω_1} (in V^P). Define $f_l: \kappa \to \kappa$ for $l = 0, 1, 2$, as in the proof of 3.6. Let

$$C = \{\delta < \kappa : \delta \text{ a limit cardinal and } i < \delta \Rightarrow f_1(i) < \delta\};$$

it is closed unbounded. So for some stationary set of $\lambda \in C$, there is an elementary embedding $j: V \to M$, with critical point λ, $H(j(f)(\lambda)) \subseteq M$. We continue as in 3.6.

As in 3.6 for each such λ, in $V^{\mathrm{Levi}(\aleph_1, <\lambda)}$, $\langle A_i : i < \lambda \rangle$ is semi-proper. Now if $N \prec (H(\beth_3(\kappa)), \in)$ is countable, $\kappa \in N$, $\langle \underset{\sim}{A}_i : i < \lambda \rangle \in N$ and $(p, q) \in \mathrm{Levi}(\aleph_1, <\kappa) * \mathrm{seal}(\bar{A})$, then for some λ

(i) $\lambda \in C \cap N$,

(ii) $\Vdash_{\mathrm{Levi}(\aleph_1, <\lambda)}$ "$\langle \underset{\sim}{A}_i : i < \lambda \rangle$ is semi-proper",

(iii) $p \in \mathrm{Levi}(\aleph_1, <\lambda)$.

We can find $p_1, p \leqq p_1 \in \mathrm{Levi}(\aleph_1, <\lambda)$ and p_1 is $(N, \mathrm{Levi}(\aleph_1, \lambda))$-generic. Let $p_1 \in G_\lambda \subseteq \mathrm{Levi}(\aleph_1, <\lambda)$, G_λ generic over V. In $V[G_\lambda]$ we can find $N_1, N \prec N_1 \prec (H(\beth_3(\kappa))[G_\lambda], \in)$ and for some $i \in N_1 \cap \lambda$, $\delta \stackrel{\text{def}}{=} N \cap \delta = N_1 \cap \delta \in \underset{\sim}{A}_i[G_\lambda]$. We can find $G \subseteq \mathrm{Levi}(\aleph_1, <\kappa)$ generic over V, $G_\lambda \subseteq G$, and then find $q_1, q \leqq q_1 \in \mathrm{seal}(\bar{A}[G])$, q_1 $(N_1, \mathrm{seal}(\bar{A}))$-generic. So there is a $\mathrm{Levi}(\aleph_1, <\kappa)$ name $\underset{\sim}{q}_1$ for such q_1 and $(p_1, \underset{\sim}{q}_1)$ is a required $(N, \mathrm{Levi}(\aleph_1, <\lambda)) * \mathrm{seal}(\tilde{A}))$-semi-generic.

4.3. Conclusion. If there are $\omega + 1$ many cardinals as in 4.1, *then* $L(\mathbf{R})$ satisfies (a), (b), (c), (d) of 1.1.

Proof. By 4.2, 3.8.

Why is this a happy end? Because Martin and Steel seem to be proving that finitely many cardinals do not suffice for 1.1(c) (hence (a), (b)).

4.4. THEOREM. *If κ satisfies Definition* 4.1 *in* $L(V_\kappa)$ *and* $L(V_\kappa)^\#$ *exists*, then *all* Σ^1_3 *sets are Lebesgue measurable*.

The proof is combining what was done above with classical proofs. Steel, hearing about the previous work but not 4.4, proved this independently. (Of course, for Σ^1_{n+3} we need n cardinals, λ below κ satisfying $\mathrm{Pr}_b(\lambda)$.)

The proof is different than for 3.2; one proves by induction on n that Σ^1_{n+4}-truth is absolute between forcing extensions by partial orders of size $<$ the least λ with $\mathrm{Pr}_b(\lambda)$. Here $L(V_\kappa)^\#$ is used. Without it we get only that Σ^1_{n+3}-truth is absolute.

Note

4.5. CLAIM. Really the inaccessibility can be replaced by $2^{2^{\kappa_l}} < \kappa_{l+1}$ (with $\kappa_{-1} = \aleph_2$).

4.6. CLAIM. We can prove our theorems also from a (strongly) compact cardinal.

4.7. REMARK. For more on $(*)_\kappa$ see Woodin [W3], where in an equivalent form (towers) they are used to produce strange forcing notions, which were previously produced only in models of *AD*.

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